5.3 Definite Integrals and Antiderivatives

Properties of Definite Integrals

Table 5.3 Rules for Definite Integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero:* $\int_a^a f(x)\,dx = 0$ — Also a definition

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ — $g = 0$

EXAMPLE 1 Using the Rules for Definite Integrals

Suppose

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Find each of the following integrals, if possible.

(a) $\displaystyle\int_{4}^{1} f(x)\,dx$ **(b)** $\displaystyle\int_{-1}^{4} f(x)\,dx$ **(c)** $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx$

(d) $\displaystyle\int_{0}^{1} f(x)\,dx$ **(e)** $\displaystyle\int_{-2}^{2} h(x)\,dx$ **(f)** $\displaystyle\int_{-1}^{4} [f(x) + h(x)]\,dx$

EXAMPLE 2 Finding Bounds for an Integral

Show that the value of $\int_0^1 \sqrt{1 + \cos x}\, dx$ is less than $3/2$.

Consider, then, what happens if we take a large *sample* of n numbers from regular subintervals of the interval $[a, b]$. One way would be to take some number c_k from each of the n subintervals of length

$$\Delta x = \frac{b-a}{n}.$$

The average of the n sampled values is

$$\begin{aligned}\frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} &= \frac{1}{n}\cdot\sum_{k=1}^{n} f(c_k)\\ &= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k) \qquad \frac{1}{n}=\frac{\Delta x}{b-a}\\ &= \frac{1}{b-a}\cdot\sum_{k=1}^{n} f(c_k)\Delta x.\end{aligned}$$

DEFINITION Average (Mean) Value

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 3 Applying the Definition

Find the average value of $f(x) = 4 - x^2$ on $[0, 3]$. Does f actually take on this value at some point in the given interval?

THEOREM 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXPLORATION 2 Finding the Derivative of an Integral

Group Activity Suppose we are given the graph of a continuous function f, as in Figure 5.27.

1. Copy the graph of f onto your own paper. Choose any x greater than a in the interval $[a, b]$ and mark it on the x-axis.
2. Using only *vertical line segments,* shade in the region between the graph of f and the x-axis from a to x. (Some shading might be below the x-axis.)
3. Your shaded region represents a definite integral. Explain why this integral can be written as $\int_a^x f(t)\,dt$. (Why don't we write it as $\int_a^x f(x)\,dx$?)
4. Compare your picture with others produced by your group. Notice how your integral (a real number) depends on which x you chose in the interval $[a, b]$. The integral is therefore a *function of* x on $[a, b]$. Call it F.
5. Recall that $F'(x)$ is the limit of $\Delta F/\Delta x$ as Δx gets smaller and smaller. Represent ΔF in your picture by drawing *one more vertical shading segment* to the right of the last one you drew in step 2. ΔF is the (signed) *area* of your vertical segment.
6. Represent Δx in your picture by moving x to beneath your newly-drawn segment. That small change in Δx is the *thickness* of your vertical segment.
7. What is now the *height* of your vertical segment?
8. Can you see why Newton and Leibniz concluded that $F'(x) = f(x)$?

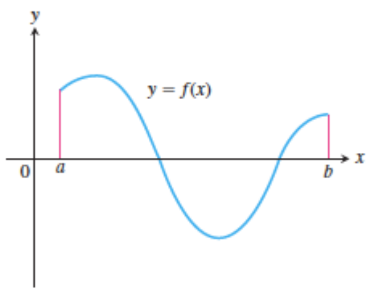

Figure 5.27 The graph of the function in Exploration 2.

area of rectangle → ΔF

$\Delta x \cdot f(x)$

$$F'(x) = \frac{\Delta F}{\Delta x} = \frac{\Delta x \cdot f(x)}{\Delta x} = f(x)$$

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt$$

$$= f(x)$$

If all went well in Exploration 2, you concluded that the derivative with respect to x of the integral of f from a to x is simply f. Specifically,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This means that the integral is an *antiderivative* of f, a fact we can exploit in the following way.

If F is any antiderivative of f, then

$$\int_a^x f(t)\,dt = F(x) + C$$

for some constant C. Setting x in this equation equal to a gives

$$\begin{aligned}\int_a^a f(t)\,dt &= F(a) + C\\ 0 &= F(a) + C\\ C &= -F(a).\end{aligned}$$

Putting it all together,

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

EXAMPLE 4 Finding an Integral Using Antiderivatives

Find $\int_0^{\pi} \sin x \, dx$ using the formula $\int_a^x f(t)\, dt = F(x) - F(a)$.

top limit lower limit

① What is the antiderivative of sin x?

$-\cos x \Big|_0^{\pi}$

② $-\cos(\pi) - (-\cos(0))$

$(1) - (-1)$

$= 2$

$w(0) = 150$

$w(24) = ?$

$$150 + \int_0^{24} \frac{1}{75}\left(600 + 20t + t^2\right) dt$$

$$150 + \frac{1}{75}\left(600t + 10t^2 + \frac{1}{3}t^3\right)\Bigg|_0^{24}$$

$=$

$$\int_1^4 f'(x)\,dx = 6.2$$

$\downarrow$

$$f(4) - f(1) = 6.2$$

$$f(4) - 3 = 6.2$$

$$f(4) = 9.2$$